Commun. Math. Phys. 132, 103–118 (1990)

Communications in
Mathematical
Physics

On the Measure of the Spectrum for the Almost Mathieu Operator*

J. Avron[1], P. H. M. v. Mouche[2], and B. Simon[3]

[1] Department of Physics, Technion, Haifa, Israel
[2] Landbouwhogschool Wageningen, Vakgroep Wiskunde, Wageningen, The Netherlands
[3] Division of Physics, Mathematics, and Astronomy, California Institute of Technology, Pasadena, CA 91125, USA

Dedicated to Res Jost and Arthur Wightman

Abstract. We obtain partial results on the conjecture that for the almost Mathieu operator at irrational frequency, α, the measure of the spectrum, $S(\alpha, \lambda, \theta) = |4-2|\lambda||$. For $|\lambda| \neq 2$ we show that if α_n is rational and $\alpha_n \to \alpha$ irrational, then $S_+(\alpha_n, \lambda, \theta) \to |4-2|\lambda||$.

1. Introduction

In this paper we will discuss the almost Mathieu operator, also called Harper's equation. This is the operator, $h_{\alpha,\lambda,\theta}$ on $l^2(\mathbb{Z})$ defined by

$$h_{\alpha,\lambda,\theta} = h_0 + v\,, \qquad (h_0 u)(n) = u(n+1) + u(n-1)\,,$$
$$(vu)(n) = \lambda \cos(2\pi\alpha n + \theta) u(n)\,,$$

where λ, α, θ are real parameters. This is the simplest of almost periodic Jacobi matrices and there has been considerable literature studying it [1, 2, 4–6, 14, 17, 18].

We will be interested in $S(\alpha, \lambda, \theta)$, the Lebesgue measure of the spectrum $\sigma(h_{\alpha,\lambda,\theta})$. It is a fundamental result (e.g. [2]) that for α irrational, S is independent of θ for α, λ fixed but this is not true if α is rational. In that case we define $S_\pm(\alpha, \lambda)$ to be the Lebesgue measure of $\sigma_\pm(\alpha, \lambda)$ where

$$\sigma_-(\alpha,\lambda) = \bigcap_\theta \sigma(\alpha,\lambda,\theta)\,, \qquad \sigma_+(\alpha,\lambda) = \bigcup_\theta \sigma(\alpha,\lambda,\theta)\,.$$

As explained in [2], $\bigcup_\theta \sigma(\alpha, \lambda, \theta)$ is the more natural object in that it has a set theoretic continuity in α.

---

* Research partially supported by U.S. NSF grant number DMS-8801918 and by BSF under grant number 88-00325