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Localization for a Class of One Dimensional Quasi-Periodic Schrödinger Operators

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Dedicated to Res Jost and Arthur Wightman with respect and affection

Abstract. We prove for small ε and α satisfying a certain Diophantine condition the operator

$$H = -\varepsilon^2 \Delta + \frac{1}{2\pi} \cos 2\pi(j\alpha + \theta) \qquad j \in \mathbb{Z}$$

has pure point spectrum for almost all θ. A similar result is established at low energy for $H = -\frac{d^2}{dx^2} - K^2(\cos 2\pi x + \cos 2\pi(\alpha x + \theta))$ provided K is sufficiently large.

1. Introduction

In this paper we shall study some of the spectral properties of the operator

$$H_c(\theta) = -\frac{d^2}{dx^2} + K^2 v(x, \theta) \tag{1.1}$$

acting on $L_2(R)$, where

$$v(x, \theta) = -\cos 2\pi x - \cos 2\pi(\alpha x + \theta). \tag{1.2}$$

We shall also study its finite difference approximation on $l_2(\mathbb{Z})$ given by

$$H(\theta) = -\varepsilon^2 \Delta + v(j, \theta) = -\varepsilon^2 \Delta + \frac{1}{2\pi} \cos 2\pi(\alpha j + \theta). \tag{1.3}$$

In one dimension the finite difference Laplacian has matrix elements $\Delta_{ij} = 1$ if $|i - j| = 1$ and $\Delta_{ij} = 0$ otherwise. When α is rational the spectra of H and H_c

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