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On Positive Multi-Lump Bound States of Nonlinear Schrödinger Equations under Multiple Well Potential

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Abstract. In this paper, we first construct multi-lump (nonlinear) bound states of the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\Delta\psi + V\psi - \gamma|\psi|^{p-1}\psi$$

for sufficiently small $\hbar > 0$, in which sense we call them "semiclassical bound states." We assume that $1 \leqq p < \infty$ for $n = 1, 2$ and $1 \leqq p < 1 + 4/(n-2)$ for $n \geqq 3$, and that V is in the class $(V)_a$ in the sense of Kato for some a. For any finite collection $\{x_1, \ldots, x_N\}$ of nondegenerate critical points of V, we construct a solution of the form $e^{-iEt/\hbar}v(x)$ for $E < a$, where v is real and it is a small perturbation of a sum of one-lump solutions concentrated near $x_1, \ldots, x_N$ respectively. The concentration gets stronger as $\hbar \to 0$. And we also prove these solutions are positive, and unstable with respect to perturbations of initial conditions for possibly smaller $\hbar > 0$. Indeed, for each such collection of critical points we construct 2^{N-1} distinct unstable bound states which may have nodes in general, and the above positive bound state is just one of them.

1. Introduction

In [W.a] and [FW.a], the following nonlinear Schrödinger equation (abbreviated as NLS) on $\mathbb{R}^n$,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\Delta\psi + V\psi - \gamma|\psi|^{p-1}\psi \tag{1}$$

was proposed to study stabilizing linear modes concentrated near local minima for sufficiently small $\hbar > 0$ for potentials bounded below. In [FW.a], Floer and Weinstein (Alan) proved the existence of solutions of (1) for sufficiently small $\hbar > 0$ for bounded potentials, which are localized near each given nondegenerate critical point of V for all time; in fact, solutions of the form $e^{-iEt/\hbar}v(x)$. In [O1], the present author generalized their existence result to arbitrary potentials in the class $(V)_a$.