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The Wulff Construction and Asymptotics of the Finite Cluster Distribution for Two-Dimensional Bernoulli Percolation

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Abstract. We consider two-dimensional Bernoulli percolation at density $p > p_c$ and establish the following results:

1. The probability, $P_N(p)$, that the origin is in a *finite* cluster of size N obeys

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \log P_N(p) = -\frac{\omega(p)\sigma(p)}{\sqrt{P_\infty(p)}},$$

where $P_\infty(p)$ is the infinite cluster density, $\sigma(p)$ is the (zero-angle) surface tension, and $\omega(p)$ is a quantity which remains positive and finite as $p \downarrow p_c$. Roughly speaking, $\omega(p)\sigma(p)$ is the minimum surface energy of a "percolation droplet" of unit area.

2. For all supercritical densities $p > p_c$, the system obeys a microscopic Wulff construction: Namely, if the origin is conditioned to be in a finite cluster of size N, then with probability tending rapidly to 1 with N, the shape of this cluster—measured on the scale $\sqrt{N}$—will be that predicted by the classical Wulff construction. Alternatively, if a system of finite volume, N, is restricted to a "microcanonical ensemble" in which the infinite cluster density is below its usual value, then with probability tending rapidly to 1 with N, the excess sites in finite clusters will form a single large droplet, which—again on the scale $\sqrt{N}$—will assume the classical Wulff shape.

1. Introduction

We consider Bernoulli bond percolation on the square lattice in which bonds are independently occupied with density p and vacant with density $1-p$. This model is known to have a phase transition at density $p_c = 1/2$, below which the occupied clusters are finite with probability one (w.p. 1) and above which there is a unique

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