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Ising Model on the Generalized Bruhat-Tits Tree

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Abstract. The partition function and the correlation functions of the Ising model on the generalized Bruhat-Tits tree are calculated. We computed also the averages of these correlation functions when the corresponding vertices are attached to the boundary of the generalized Bruhat-Tits tree.

1. Introduction

The Ising model on the Cayley tree turns out to be very interesting [1, 2]. The Cayley tree T is manifestly determined to be a connected infinite graph with no loops, each vertex of T being connected with exactly $p+1$ nearest neighbour vertices by links. If p is a prime number, the Cayley tree is called the Bruhat-Tits tree. The branch B_z is defined to be a connected subtree with the only boundary vertex z of the graph $T\backslash B_z$ in the interior of T. By definition the branch contains no cycles. Let us introduce the generalized Bruhat-Tits tree F_g. It consists of a finite connected graph F_g^R with g independent loops, which is called a *reduced* graph, the branches B_x, $x \in F_g^R$, and each vertex is connected by links with exactly $p+1$ nearest neighbours (for every link, two endpoints of which are identified with a vertex, we include the vertex itself twice into the number of its nearest neighbours). If the vertex $x \in F_g^R$ has only one nearest neighbour $y \in F_g^R$, $x \neq y$, then p branches B_x and the link $[x, y]$ form the branch B_y. Hence instead of the reduced graph F_g^R we may consider the reduced graph $F_g^R \backslash [x, y]$. From now on F_0^R is merely a single vertex and $p+1$ branches should be added to this vertex in order to construct the Bruhat-Tits tree $F_0 = T$, for $g > 0$ each vertex $x \in F_g^R$ has $2 \leqq n(x) \leqq p+1$ nearest neighbours in F_g^R and $b(x) = p+1-n(x)$ branches should be added to x in order to construct the generalized Bruhat-Tits tree F_g. Due to [3–5] the Bruhat-Tits tree $T \equiv F_0$ may be interpreted as the coset space $PGL(2, \mathbb{Q}_p)/PGL(2, \mathbb{Z}_p)$, where $PGL(2, \mathbb{K})$ is the group of fractional linear transformations of the projective line $P^1(\mathbb{K})$ over a ring $\mathbb{K}$ (we deal with the field of p-adic numbers $\mathbb{Q}_p$ and with the ring of the p-adic integers $\mathbb{Z}_p$). The element of $GL(2, \mathbb{Q}_p)$ is called hyperbolic if it has eigenvalues which p-adic norms are different. A Schottky group Γ_g is a free subgroup of $PGL(2, \mathbb{Q}_p)$ with g generators, all non-unit elements of which are hyperbolic. Usually the generalized Bruhat-Tits tree F_g may be interpreted as a coset space T/Γ_g, where Γ_g is some Schottky group [4–6].

Since the configuration σ takes the values ± 1 the Ising model action may be rewritten in the form

$$\beta \sum_{|x-y|=1} \sigma(x)\sigma(y) = \beta N_1 - \beta/2 \sum_{|x-y|=1} (\sigma(x)-\sigma(y))^2,$$