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On the Thermodynamic Formalism for the Gauss Map

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Abstract. We study the generalized transfer operator $\mathscr{L}_\beta f(z) = \sum_{n=1}^{\infty} \left(\frac{1}{z+n}\right)^{2\beta} \times f(1/(z+n))$ of the Gauss map $Tx = (1/x) \bmod 1$ on the unit interval. This operator, which for $\beta = 1$ is the familiar Perron–Frobenius operator of T, can be defined for $\operatorname{Re}\beta > \frac{1}{2}$ as a nuclear operator either on the Banach space $A_\infty(D)$ of holomorphic functions over a certain disc D or on the Hilbert space $\mathscr{H}^{(2)}_{\operatorname{Re}\beta}(H_{-1/2})$ of functions belonging to some Hardy class of functions over the half plane $H_{-1/2}$. The spectra of $\mathscr{L}_\beta$ on the two spaces are identical. On the space $\mathscr{H}^{(2)}_{\operatorname{Re}\beta}(H_{-1/2})$ $\mathscr{L}_\beta$ is isomorphic to an integral operator $\mathscr{K}_\beta$ with kernel the Bessel function $\mathfrak{J}_{2\beta-1}(2\sqrt{st})$ and hence to some generalized Hankel transform. This shows that $\mathscr{L}_\beta$ has real spectrum for real $\beta > \frac{1}{2}$. On the space $A_\infty(D)$ the operator $\mathscr{L}_\beta$ can be analytically continued to the entire β-plane with simple poles at $\beta = \beta_k = (1-k)/2$, $k = 0, 1, 2, \ldots$ and residue the rank 1 operator $\mathscr{N}^{(k)} f = \frac{1}{2}(1/k!) f^{(k)}(0)$. From this similar analyticity properties for the Fredholm determinant $\det(1 - \mathscr{L}_\beta)$ of $\mathscr{L}_\beta$ and hence also for Ruelle's zeta function follow. Another application is to the function $\zeta_M(\beta) = \sum_{n=1}^{\infty} [n]^\beta$, where $[n]$ denotes the irrational $[n] = (n + (n^2+4)^{1/2})/2$. $\zeta_M(\beta)$ extends to a meromorphic function in the β-plane with the only poles at $\beta = \pm 1$ both with residue 1.

1. Generalized Transfer Operators for the Gauss Map

If $I = [0, 1]$ denotes the unit interval in $\mathbb{R}$ the Gauss (or continued fraction-)map $T: [0,1] \to [0,1]$ is defined as

$$Tx = \begin{cases} 1/x \bmod 1 & x \neq 0 \\ 0 & x = 0 \end{cases}. \tag{1}$$

From ergodic theory for general hyperbolic systems $T: M \to M$ it is known [Bo], [Ru1] that systems like the Gauss map allow for a description in terms of symbolic dynamics $\pi: F^{\mathbb{Z}_+} \to M$ with an alphabet F and a transition matrix $\mathbb{A} = (\mathbb{A}_{\sigma,\sigma'})$, $\sigma, \sigma' \in F$, defined through a Markov partition $\mathscr{A} = (\mathscr{O}_\sigma)_{\sigma \in F}$. This way T gets