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Topological Gauge Theories and Group Cohomology

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Abstract. We show that three dimensional Chern-Simons gauge theories with a compact gauge group G (not necessarily connected or simply connected) can be classified by the integer cohomology group $H^4(BG,\mathbf{Z})$. In a similar way, possible Wess-Zumino interactions of such a group G are classified by $H^3(G,\mathbf{Z})$. The relation between three dimensional Chern-Simons gauge theory and two dimensional sigma models involves a certain natural map from $H^4(BG,\mathbf{Z})$ to $H^3(G,\mathbf{Z})$. We generalize this correspondence to topological "spin" theories, which are defined on three manifolds with spin structure, and are related to what might be called $\mathbf{Z}_2$ graded chiral algebras (or chiral superalgebras) in two dimensions. Finally we discuss in some detail the formulation of these topological gauge theories for the special case of a finite group, establishing links with two dimensional (holomorphic) orbifold models.

1. Introduction

Topological gauge field theories in three dimensions are related in an interesting way to two dimensional mathematical physics [1] and are interesting as well for their purely geometrical content. One of the key ingredients in formulating three dimensional topological gauge theories is the Chern-Simons action functional. Thus, let M be an oriented three manifold, G a compact gauge group, Tr an invariant quadratic form on the Lie algebra of G, and A a connection on a G bundle E. If E is trivial, the connection A can be regarded as a Lie algebra valued one form, and we can define the Chern-Simons functional by the familiar formula

$$S(A)=\frac{k}{8\pi^2}\int_M \mathrm{Tr}(A\wedge dA+\tfrac{2}{3}A\wedge A\wedge A)\,. \tag{1.1}$$

One can then use this functional as the Lagrangian of a quantum field theory. In this paper we use a normalization in which the path integral reads

$$Z(M)=\int \mathscr{D}A\, e^{2\pi i S(A)}\,. \tag{1.2}$$

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