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Construction of Solutions with Exactly k Blow-up Points for the Schrödinger Equation with Critical Nonlinearity

Frank Merle

Centre de Mathématiques Appliquées, Ecole Normale Supérieure, 45, rue d'Ulm, F-75230 Paris, Cedex 05, France

Abstract. We consider the nonlinear Schrödinger equation:

$$i\partial u/\partial t = -\Delta u - |u|^{4/N}u \quad \text{and} \quad u(0,.) = \varphi(.), \tag{1}$$

where $u\colon [0,T)\times \mathbb{R}^N \to \mathbb{C}$. For any given points $x_1, x_2, \ldots, x_k$ in $\mathbb{R}^N$, we construct a solution of Eq. (1), $u(t)$, which blows up in a finite time T at exactly $x_1, x_2, \ldots, x_k$. In addition, we describe the precise behavior of the solution $u(t)$ when $t \to T$, at the blow-up points $\{x_1, x_2, \ldots, x_k\}$ and in $\mathbb{R}^N - \{x_1, x_2, \ldots, x_k\}$.

I. Introduction and Main Results

In the present paper, we consider the Schrödinger equation:

$$i\partial u/\partial t = -\Delta u - |u|^{p-1}u \quad \text{and} \quad u(0,.) = \varphi(.), \tag{1}$$

where Δ is the Laplace operator on $\mathbb{R}^N$, $u\colon [0,T)\times \mathbb{R}^N \to \mathbb{C}$, $p = 1 + 4/N$, and $\varphi \in H^1(\mathbb{R}^N)$. More precisely, we say that $u(.)$ is a solution of Eq. (1) on $[0,T)$ if $\forall t \in [0,T)$,

$$u(t) = S(t)\varphi + i\int_0^t S(t-s)\{|u(s)|^{4/N}u(s)\}\,ds,$$

where $S(.)$ is the group with infinitesimal generator $i\Delta$ (the Schrödinger group) and for each t, $u(t)$ denotes the function $x \to u(t,x)$.

For $p \in (1, 2^* - 1)$ (where $2^* = 2N/(N-2)$ if $N > 2$, otherwise $2^* = +\infty$), it is well known that Eq. (1) has a unique solution $u(t)$ in H^1 and there exists $T > 0$ such that $\forall t \in [0,T)$, $u(t) \in H^1$ and either $T = +\infty$ or $\lim_{t\to T} \|u(t)\|_{H^1} = +\infty$ (see Ginibre and Velo [4, 5], Kato [7]). Furthermore, we have $\forall t \in [0,T)$,

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2}, \tag{2}$$

$$E(u(t)) = (1/2)\|\nabla u(t)\|_{L^2}^2 - (1/(p+1))\int |u(t,x)|^{p+1}\,dx = E(\varphi). \tag{3}$$