Commun. Math. Phys. 128, 319–332 (1990)

Communications in
Mathematical
Physics

Classical Solutions of the Chiral Model, Unitons, and Holomorphic Vector Bundles

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Abstract. This paper deals with classical solutions of the $SU(2)$ chiral model on $\mathbb{R}^2$, and of a generalized chiral model on $\mathbb{R}^{2+1}$. Such solutions are shown to correspond to certain holomorphic vector bundles over minitwistor space. With an appropriate boundary condition, the solutions (called 1-unitons in [9]) correspond to bundles over a compact 2-dimensional complex manifold, and the problem becomes one of algebraic geometry.

1. Introduction

Minitwistor space $T\mathbb{P}_1$ is a 2-dimensional complex manifold which was used by Hitchin [5, 6] in the construction of monopoles on 3-dimensional Euclidean space. The solutions of the Bogomolny equations for monopoles on $\mathbb{R}^3$ correspond to certain holomorphic vector bundles over $T\mathbb{P}_1$. However, by imposing a different "reality" condition on such vector bundles, one can generate the solutions of the hyperbolic version of the Bogomolny equations, i.e. solutions which live on $(2+1)$-dimensional space-time $\mathbb{R}^{2+1}$. These equations form an integrable hyperbolic system, and they include, as special cases, such well-known soliton equations as the sine-Gordon, Korteweg-de Vries and nonlinear Schrödinger equations. So solutions of these correspond to holomorphic vector bundles over $T\mathbb{P}_1$; see [12] for more details.

The purpose of this paper is to deal with holomorphic vector bundles which extend to a certain compactification $\mathbb{T}$ of $T\mathbb{P}_1$. This excludes, for example, those bundles which correspond to soliton solutions of the sine-Gordon equation. But it turns out to be the right sort of boundary condition for the chiral model in $2+1$ dimensions. The hyperbolic Bogomolny equations referred to above can be rewritten as a chiral equation with torsion term, and the aim in this paper is to describe how solutions of this chiral equation correspond to (and can be generated from) vector bundles over $T\mathbb{P}_1$ or $\mathbb{T}$.

A chiral field is a map from $\mathbb{R}^{2+1}$ into a Lie group G, satisfying a certain nonlinear equation. In the case of static (time-independent) fields, one has a map