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L^2-Index Formulae for Perturbed Dirac Operators

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Abstract. The Callias index theorem is generalized from the Euclidean case to certain spin manifolds with warped ends, making use of certain index-preserving deformations.

0. Introduction

Physical considerations led C. Callias to the following open space index theorem (cf. [C]):

Theorem 0.1. *Let Σ be the spinor space over $\mathbb{R}^n, n$ odd, and D the Dirac operator on $C^\infty(\mathbb{R}^n, \Sigma \otimes \mathbb{C}^m)$. Let L be the perturbation of D by $\sqrt{-1}\,\mathrm{Id} \otimes \Phi$, where $\Phi \in C^\infty(\mathbb{R}^n, \mathrm{End}(\mathbb{C}^m))$ is Hermitian, asymptotically homogeneous of degree* 0, *and Φ^2 is positive outside some compact set. Then L is a Fredholm elliptic differential operator, and if U is the unitarization of Φ at infinity,* i.e., *$U = |\Phi|^{-1}\Phi$ outside a compact set, one has*

$$L^2\text{-index}(L) = \frac{1}{2\left(\frac{n-1}{2}\right)!}\left(\frac{\sqrt{-1}}{8\pi}\right)^{(n-1)/2} \lim_{R \to \infty} \int_{\mathbb{S}_R^{n-1}} \operatorname{tr} U(dU)^{n-1}. \tag{0.2}$$

In (0.2), $\mathbb{S}_R^{n-1}$ stands for the sphere centered at the origin and of radius R in $\mathbb{R}^n$. As remarked by H. Moscovici, the formula (0.2) can be rewritten as follows:

$$L^2\text{-index}(L) = \operatorname{ch}(V_+)[\mathbb{S}_\infty^{n-1}]. \tag{0.3}$$

The right-hand side of (0.3) represents the evaluation on $\mathbb{S}_\infty^{n-1}$, the sphere at infinity in $\mathbb{R}^n$, of the Chern character of the subbundle V_+ of $\mathbb{C}^m$ over $\mathbb{S}_\infty^{n-1}$ given by $V_+ = \{U = \mathrm{Id}\}$.

While Callias' result and method of proof attracted a lot of interest (see [Bo–S], [S]), there is no direct generalization of (0.1) that we know of. In this paper we attempt a generalization of Theorem 0.1 based on the observation (0.3) to a

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