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Ising Models on the Lobachevsky Plane

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Abstract. We consider the Ising model on a lattice which is the orbit of a discrete cocompact group acting on the hyperbolic plane. For large values of the inverse temperature we construct an uncountable number of mutually singular Gibbs states.

1. Introduction

The goal of this paper is to study some Ising models on the Lobachevsky plane. Before discussing the motivation and the formulation of the results we shall give some necessary definitions.

Let $\mathbb{H}$ be the Lobachevsky plane and let G be a finitely generated co-compact group of isometries of $\mathbb{H}$. We shall build our model on the Cayley graph $\hat{\mathscr{G}}$ of G. The graph $\hat{\mathscr{G}}$ is embedded in $\mathbb{H}$ in the following way. Choose a convex finite-sided geodesic polygon A which is a fundamental domain for G acting on $\mathbb{H}$. According to Poincare′'s theorem on fundamental polygons, the set of isometries G_0 which identify the sides of A is a set of generators of G. Fix a point $0 \in \operatorname{Int} A$. The vertices of $\hat{\mathscr{G}}$ are the points $g0$, $g \in G$. Join $g0$ to $g'0$ by an edge whenever $g^{-1}g' \in G_0$.

We consider spin configurations $\varphi = \{\varphi(g)\}_{g \in G}$ on $\hat{\mathscr{G}}$, where $\varphi(g)$ takes the values ± 1 at the vertex $g0 \in \hat{\mathscr{G}}$. Sites $g0$, $g'0$ have a common bond, written $\langle g, g' \rangle$, exactly when $g0$, $g'0$ are joined by an edge. The Ising ferromagnetic Hamiltonian at the inverse temperature β has the form

$$H_\beta(\varphi) = -\beta \sum_{\langle g, g' \rangle} \varphi(g)\varphi(g'). \tag{1}$$

For such models it is impossible to define the notion of free energy because for natural domains like balls the number of points belonging to the boundary is proportional to the number of points in the whole domain. However, the notion of a Gibbs state can be introduced in an unambiguous way through the DLR conditions. Namely, let Φ be the space of all configurations $\varphi = \{\varphi(g)\}_{g \in G}$, let $\mathscr{F}$ be the σ-algebra of Borel subsets of Φ and let P be some probability measure defined on $\mathscr{F}$.