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Modular Transformations of $SU(N)$ Affine Characters and Their Commutant

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Abstract. We describe the algebra of matrices commuting with the action of the modular group on characters of $SU(N)_k$ integrable representations. Using methods of finite quantum mechanics we find a canonical basis for this commutant over $\mathbb{C}$ and prove the existence of an equivalent basis over $\mathbb{Q}$ with integral matrix elements. A final section is devoted to the case of $SU(3)$.

1. Introduction

One of the goals in conformal field theory is to classify all rational models. Automorphisms of the algebra of fusion rules and conformal embeddings enable one to construct new modular invariant partition functions from previously known ones. The Wess-Zumino-Witten models associated to simple Lie groups are among the most tractable examples and are believed to be the basic bricks for the construction of all rational theories. However, even in this case it has only been possible to exhibit an exhaustive list of modular invariants in a limited number of instances ($SU(2)_k$ and $SU(N)_1$ and corresponding coset models being the most conspicuous) and the proof of completeness (of arithmetical nature) used methods radically different from the above mentioned ones. In the present work we describe partial results pertaining to a more general situation following the arithmetical path. It is not unlikely that a number of arguments collected in the next sections appear in one form or another in the mathematical literature. We thought it however useful to present them in some detail for a nonexpert reader.

For a simple Lie group and a given level k, the associated Kac-Moody algebra admits only a finite number of integrable representations, with (restricted) characters $\chi_\lambda(\tau)$, indexed by λ, depending on a complex variable τ in the upper half plane $\operatorname{Im}\tau > 0$, which carry a unitary representation of the modular group acting on τ [1]. The space of states of the theory decomposes as a sum $\bigoplus_{\lambda,\lambda'} Z_{\lambda,\lambda'}\mathscr{H}_\lambda \otimes \mathscr{H}_{\lambda'}$,

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