Commun. Math. Phys. 127, 529–553 (1990)

Communications in Mathematical Physics

Einstein Metrics on S^3, R^3 and R^4 Bundles

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Abstract. Starting from a $4n$-dimensional quaternionic Kähler base space, we construct metrics of cohomogeneity one in $(4n+3)$ dimensions whose level surfaces are the S^2 bundle space of almost complex structures on the base manifold. We derive the conditions on the metric functions that follow from imposing the Einstein equation, and obtain solutions both for compact and non-compact $(4n+3)$-dimensional spaces. Included in the non-compact solutions are two Ricci-flat 7-dimensional metrics with G_2 holonomy. We also discuss two other Ricci-flat solutions, one on the R^4 bundle over S^3 and the other on an R^4 bundle over S^4. These have G_2 and Spin(7) holonomy respectively.

1. Introduction

There are many examples of homogeneous Einstein metrics to be found in the literature, but inhomogeneous examples, where there is no transitively-acting isometry group, are much rarer. In this paper, we construct examples in $4n+3$ dimensions which can be described as S^3 or R^3 bundles over quaternionic Kähler base manifolds. After reviewing some relevant properties of quaternionic Kähler spaces, in this section we then discuss the notion of the twistor space Z corresponding to a quaternionic Kähler space M [1]. This space plays a central role in the rest of the paper. In Sect. 2 we give a local discussion of our construction, including details of the local calculation of the curvature of our spaces. In Sect. 3 we consider the regularity conditions on the local metrics that ensure that they can be extended to globally-defined metrics on complete manifolds, and we apply these conditions to discuss the existence of complete Einstein metrics on compact manifolds, which we have found numerically. In Sect. 4, we consider non-compact Ricci-flat spaces, and present two exact solutions in seven dimensions. These are the same as the seven-dimensional metrics with G_2 holonomy constructed recently by using different methods [2]. In Sect. 5, we consider two more exact Ricci-flat metrics, one on the manifold $R^4 \times S^3$, with G_2 holonomy, and the other on an R^4 bundle over S^4, with Spin(7) holonomy. Again, these coincide with examples