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Periodic and Quasi-periodic Solutions of Nonlinear Wave Equations via KAM Theory

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Abstract. In this paper the nonlinear wave equation

$$u_{tt} - u_{xx} + v(x)u(x,t) + \varepsilon u^3(x,t) = 0$$

is studied. It is shown that for a large class of potentials, $v(x)$, one can use KAM methods to construct periodic and quasi-periodic solutions (in time) for this equation.

I. Introduction

This paper studies the non-linear wave equation

$$\begin{gathered} u_{tt}(x,t) - u_{xx}(x,t) + v(x)u(x,t) + \varepsilon u^3(x,t) = 0, \\ 0 \leqq x \leqq 1, \quad t \geqq 0; \quad u(0,t) = u(1,t) = 0; \quad v \in L^2[0,1]. \end{gathered} \tag{1.1}$$

We show that for a large class of potentials, v, one can construct periodic and quasi-periodic solutions for (1.1), provided ε is small, using a variant of the Kolmogorov, Arnold, Moser [KAM] scheme. The method allows one to study more general non-linear terms than the cubic term in (1.1). For a discussion of the types of non-linearities that are permitted, see Sect. 2.

The existence of solutions, periodic in time, for non-linear wave equations has been studied by many authors. (See [B] for a review of these results. [BN] contains an extensive bibliography.) A wide variety of methods have been brought to bear on the problem, ranging from bifurcation theory, (see for example [H]), to variational techniques, pioneered by Rabinowitz [R], to ideas which exploit the hamiltonian structure of the problem.

The KAM techniques are somewhat complementary to these approaches. They are local methods in that they can only be applied if ε is small, (or equivalently to construct solutions $u(x,t)$ of small norm) whereas the variational methods often yield global results. On the other hand the variational techniques place very strong restrictions on the allowed periods of the solutions. The period, in time, must be

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