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Ribbon Graphs and Their Invariants Derived from Quantum Groups

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Abstract. The generalization of Jones polynomial of links to the case of graphs in R^3 is presented. It is constructed as the functor from the category of graphs to the category of representations of the quantum group.

1. Introduction

The present paper is intended to generalize the Jones polynomial of links and the related Jones-Conway and Kauffman polynomials to the case of graphs in R^3.

Originally the Jones polynomial was defined for links of circles in R^3 via an astonishing use of von Neumann algebras (see [Jo]). Later on it was understood that this and related polynomials may be constructed using the quantum R-matrices (see, for instance, [Tu_1]). This approach enables one to construct similar invariants for coloured links, i.e. links each of whose components is provided with a module over a fixed algebra (see [Re_1], where the role of the algebra is played by the quantized universal enveloping algebra $U_q(G)$ of a semisimple Lie algebra G).

The Jones polynomial has been also generalized in another direction: in generalization of links of circles one considers the so-called tangles which are links of circles and segments in the 3-ball, where it is assumed that ends of segments lie on the boundary of the ball. Technically it is convenient to replace the ball by the strip $R^2 \times [0, 1]$, which enables one to distinguish the top and bottom endpoints of the tangle. The corresponding "Jones polynomial" of a coloured tanlge is a linear operator $V_1 \otimes \ldots \otimes V_k \to V^1 \otimes \ldots \otimes V^\ell$ where $V_1, \ldots, V_k$ (respectively $V^1 \otimes \ldots \otimes V^\ell$) are the modules associated with the segments incident to bottom (respectively top) endpoints of the tangle. Here the language of categories turns out to be very fruitful. The tangles considered up to isotopy are treated as morphisms of the "category of tangles." The generalized Jones polynomial is a covariant functor from this category to the category of modules (see [Tu_2, Re_2]).

Definitions of Jones-type polynomials for embedded graphs in R^3 have been given by several authors [KV, Ye] but the subject still remains open. It was clear from the very beginning that the graph should be provided with thickening, i.e.