Commun. Math. Phys. 126, 409–420 (1989)

Communications in Mathematical Physics

Some Comments on Chern-Simons Gauge Theory

T. R. Ramadas[1], I. M. Singer, and J. Weitsman

Department of Mathematics, M.I.T., Cambridge, MA 02139, USA

Abstract. Following M. F. Atiyah and R. Bott [AB] and E. Witten [W], we consider the space of flat connections on the trivial $SU(2)$ bundle over a surface M, modulo the space of gauge transformations. We describe on this quotient space a natural hermitian line-bundle with connection and prove that if the surface M is now endowed with a complex structure, this line bundle is isomorphic to the determinant bundle. We show heuristically how path-integral quantisation of the Chern-Simons action yields holomorphic sections of this bundle.

1. Introduction

In [W], Witten studied a $2+1$ dimensional quantum Yang-Mills theory, with an action consisting purely of the Chern-Simons term,

$$CS(\mathbf{A}) = \frac{1}{4\pi} \int Tr\left(\mathbf{A}d\mathbf{A} + \frac{2}{3}\mathbf{AAA}\right).$$

He obtained the Jones polynomials of knots on S^3 and their extensions to other 3-manifolds as expectation values of Wilson loop functionals. A key point was the identification of the quantum state space as the space of holomorphic sections of a line bundle.

We first describe this line bundle from an algebraic point of view. Let M denote a compact 2-manifold without boundary (with genus $\mathbf{g} \geqq 3$ – the other cases can be treated with analogous results), $\mathscr{A}$ the space of connections on the trivial $SU(2)$ bundle on M, $\mathscr{A}_F$ the space of flat connections, $\mathscr{A}_F^s$ the space of irreducible flat connections, and $\mathscr{G}$ the group of gauge transformations. Then it is well-known that $\mathscr{A}_F^s/\mathscr{G}$ is in a natural way a symplectic manifold. A choice of conformal structure M_c on M endows $\mathscr{A}_F^s/\mathscr{G}$ with a compatible Kähler structure, and it can be

[1] On leave from the School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

I.M.S. and T.R.R. supported by DOE grant DE-FG02-88ER25066. J.W. supported by NSF Mathematical Sciences post-doctoral research scholarship 8807291