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Stratifying Monopoles and Rational Maps

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Abstract. A stratification of the moduli space of monopoles and of the space of rational maps into a flag variety is presented. It is shown that the map associating a rational map to a monopole preserves these strata. These strata explain some problems in the intepretation of the parameters of the moduli space in terms of superpositions of fundamental monopoles. This interpretation is not valid on the individual strata. The space of fundamental monopoles is described and shown to be the same as the corresponding space of rational maps.

1. Introduction

Recently Hurtubise (1988) showed that the moduli space of framed $SU(N)$ monopoles, with Higgs field at infinity having distinct eigenvalues, is isomorphic to the space of based rational maps of one dimensional complex projective space, $\mathbb{P}_1$, into the space of full flags in $\mathbb{C}^N$. This extends the work of Donaldson (1984) who proved the same result for $SU(2)$. These results are particular cases of a general conjecture of Atiyah that the moduli space $\mathscr{M}(X)$ of framed monopoles for a group K, with Higgs field at infinity taking its values in an adjoint orbit X of the group on its Lie algebra, is isomorphic to the space $\mathscr{R}(X)$ of based rational maps of $\mathbb{P}_1$ into X.

The Higgs field at infinity of a monopole defines a class m in $\pi_2(X)$ and this labels a decomposition of the moduli space into disconnected pieces $\mathscr{M}(m, X)$. A more precise statement of Atiyah's conjecture is that $\mathscr{M}(m, X)$ is isomorphic to the corresponding connected component $\mathscr{R}(m, X)$ of $\mathscr{R}(X)$. This implies, in particular, that $\mathscr{M}(m, X)$ is connected; a result that has been proven only for $SU(2)$ and $SU(3)$ (Taubes 1985).

It seems likely that the methods of Hurtubise will generalise to $SU(N)$ monopoles with any symmetry breaking at infinity. We examine what this means for the moduli space of monopoles. In particular both $\mathscr{M}(m, X)$ and $\mathscr{R}(m, X)$ have stratifications and we show that these are preserved by the mapping assigning a rational map to a monopole.

In Weinberg 1982 it was suggested that the monopole parameters could be