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Global Laurent Expansions on Riemann Surfaces*

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Abstract. We discuss global Laurent expansions for meromorphic h-forms on a compact Riemann surface of genus $g \geqq 2$. Our approach is motivated by Krichever and Novikov's work on string theory.

I. Introduction

Krichever and Novikov [KN1, 2], in their study connected to conformal field theory on a Riemann surface S of arbitrary genus g, introduced the notion of a global Laurent expansion of a meromorphic h-form on S. Their approach consists in the following. Let P_0 and P_∞ be two distinguished points on S in general position. There exists a sequence of h-forms $f_n^{(h)}, n, h \in \mathbb{Z}$, holomorphic on S except, possibly, for P_0 and P_∞, where the orders of $f_n^{(h)}$ are prescribed. The forms $\{f_n^{(h)}\}$ serve as a basis with respect to which an h-form ω holomorphic in an annulus on S (see Sect. II for the definition) can be expanded in a convergent series. In the case of $g=0$ these forms are given by $f_n^{(h)}(x) = x^{n-h}(dx)^h$. The special feature of these expansions is that they are formulated in a coordinate independent way. To the best of our knowledge, global expansion on a Riemann surface were first discussed in [BS].

The present study is concerned with a detailed analysis of the Krichever–Novikov (KN) expansions. We find an explicit representation of $f_n^{(h)}$ in terms of the Riemann theta function and prove pointwise estimates on $f_n^{(h)}$. It appears that a complete proof of convergence of the KN expansion is impossible without this explicit form of $f_n^{(h)}$. In particular, crucial for the convergence is a detailed analysis of the normalization constants occurring in $f_n^{(h)}$. This leads to a small denominator problem which has not been discussed in [KN1, 2] and which is settled here. Furthermore, we define generalized Cauchy kernels $K^{(h)}(x, y)$ which serve to generate the expansion. Using Fay's trisecant identities [F] we find closed form expressions for $K^{(h)}(x, y)$. Similar representations of $f_n^{(h)}$ and $K^{(h)}(x, y)$ can be found in the context of conformal $b-c$ systems [S], [BLMR] (this was brought to our attention by Hidenori Sonoda); our point is also to address the analytic questions. We would like to mention that there is a relation between the KN approach to the chiral algebras on Riemann surfaces and the operator formalism developed in [AGMV].

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