Commun. Math. Phys. 125, 545–563 (1989)

Communications in Mathematical Physics

Nonlinear Poisson Structures and r-Matrices

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Abstract. We introduce quadratic Poisson structures on Lie groups associated with a class of solutions of the modified Yang–Baxter equation and apply them to the Hamiltonian description of Lax systems. The formal analog of these brackets on associative algebras provides second structures for certain integrable equations. In particular, the integrals of the Toda flow on generic orbits are shown to satisfy recursion relations. Finally, we exhibit a third order Poisson bracket for which the r-matrix approach is feasible.

1. Introduction

The classical r-matrices were first introduced by E. Sklyanin in [17] and [18] as limits of their quantum counterparts. Subsequently, this has led V. G. Drinfel'd to introduce a new geometric concept, that of a Poisson Lie group [7]. The relevance of these notions in the study of classical integrable systems was recently explained in two fundamental papers of M. Semenov–Tian–Shansky [15, 16]. By abandoning the classical Yang–Baxter equation in favor of the modified Yang–Baxter equation (mYB), the result is a unification of the generalized Adler–Kostant–Symes procedure and the method of the Riemann problem [15]. Furthermore, in the second half of [15] and in [16], it was revealed that the r-matrix approach is naturally associated with a class of quadratic Poisson structures commonly referred to as the Sklyanin brackets. These quadratic Poisson structures on Lie groups (and modifications thereof [16]) are associated with skew symmetric solutions of (mYB) and give rise to a geometrical theory of Lax systems and dressing transformations. On the other hand, their formal analog on associative algebras provides an abstract version of the "second Hamiltonian structure" for equations of KdV type as conjectured by Adler [2] and proved in [9] by Gelfand and Dikii.

It is the purpose of this paper to extend the theory of Lax systems and the construction of "second Poisson structures" in [15] and [16] to a wider class of r-matrices. This will be carried out in Sects. 3 and 4 below. Instead of assuming the r-matrix $R \in \operatorname{End}\mathfrak{g}$ to be skew symmetric, we shall assume that R and $A = \frac{1}{2}(R - R^*)$ are solutions of (mYB). Here, the choice of this particular class of