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# The Ising Model and Percolation on Trees and Tree-Like Graphs

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**Abstract.** We calculate the exact temperature of phase transition for the Ising model on an arbitrary infinite tree with arbitrary interaction strengths and no external field. In the same setting, we calculate the critical temperature for spin percolation. The same problems are solved for the diluted models and for more general random interaction strengths. In the case of no interaction, we generalize to percolation on certain tree-like graphs. This last calculation supports a general conjecture on the coincidence of two critical probabilities in percolation theory.

## 1. Introduction

Consider a tree, as in Fig. 1; we use the word *tree* to mean a countable connected graph which has no loops or cycles and which is locally finite (i.e., each vertex belongs only to a finite number of edges). In the Ising model of ferromagnetism [**KS, Pr1, Big**], there is a particle at each vertex with spin either up ($+1$) or down ($-1$). Each particle interacts with its nearest neighbors in such a way as to favor alignment of the spins; we shall assume that there is no external magnetic field. At temperatures higher than a certain critical temperature, $T_c$, there is only one Gibbs state, while at temperatures below $T_c$, there are at least two. (In fact, for $T < T_c$, there are an uncountable number of extreme Gibbs states on a tree.) Clearly, adding edges and vertices to a tree can only increase its critical temperature [**Lig**, Theorem IV.1.21, p. 186]. We shall at first assume that the interaction strength is the same along all edges. Thus, $T_c$ is a measure of the number of edges per vertex "on average." Remarkably, after a scale conversion, this notion of average number of edges per vertex coincides with one which has already been studied in connection with percolation, random walks, and Hausdorff dimension [**Lyo**]. This correspondence between the Ising model and percolation, exact for trees, is only approximate for other graphs [**Bis1, Bis3**].

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