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Spectral Asymptotics for the Schrödinger Operator with Potential which Steadies at Infinity*

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Abstract. We consider the discrete spectrum of the selfadjoint Schrödinger operator $A_h = -h^2\Delta + V$ defined in $L^2(\mathbb{R}^m)$ with potential V which steadies at infinity, i.e. $V(x) = g + |x|^{-\alpha} f(1+o(1))$ as $|x| \to \infty$ for $\alpha > 0$ and some homogeneous functions g and f of order zero. Let $\mathfrak{N}_h(\lambda)$, $\lambda \geqq 0$, be the total multiplicity of the eigenvalues of A_h smaller than $M-\lambda$, M being the minimum value of g over the unit sphere S^{m-1} (hence, M coincides with the lower bound of the essential spectrum of A_h). We study the asymptotic behaviour of $\mathfrak{N}_1(\lambda)$ as $\lambda \downarrow 0$, or of $\mathfrak{N}_h(\lambda)$ as $h \downarrow 0$, the number $\lambda \geqq 0$ being fixed. We find that these asymptotics depend essentially on the structure of the submanifold of S^{m-1}, where the function g takes the value M, and generically are nonclassical, i.e. even as a first approximation $(2\pi)^m \mathfrak{N}_h(\lambda)$ differs from the volume of the set $\{(x,\xi) \in \mathbb{R}^{2m} : h^2|\xi|^2 + V(x) < M - \lambda\}$.

1. Introduction

Let $\mathfrak{A}_h \equiv -h^2\Delta + V$ be the Schrödinger operator with domain $C_0^\infty(\mathbb{R}^m)$, $m \geqq 3$. Here $h > 0$ is a constant parameter, Δ is the Laplacian, and V is a real-valued potential which is supposed to possess the following properties:

i) $V \in L^{m/2}_{\mathrm{loc}}(\mathbb{R}^m)$;
ii) V steadies at infinity, i.e. there exist two continuous real-valued functions f and g over the unit sphere S^{m-1} and a positive number α such the asymptotic relation

$$\lim_{|x| \to \infty} |x|^\alpha (V(x) - g(\hat{x})) = f(\hat{x}), \; \hat{x} \equiv x/|x| \;,$$

holds uniformly with respect to $\hat{x} \in S^{m-1}$;

Then $\mathfrak{A}_h$ is symmetric and semibounded from below in $L^2(\mathbb{R}^m)$. Denote by A_h the selfadjoint Friedrichs extension of $\mathfrak{A}_h$.

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