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The Factorization of a Polynomial Defined by Partitions

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Abstract. Polynomials whose vanishing is necessary and sufficient for the existence of primary holomorphic conformal fields are introduced, and in certain cases decomposed into linear factors.

1. Introduction

It is best to work with unordered partitions. Thus if k is a positive integer, a partition of length r of the interval $[0, k]$ is a sequence, $0 = k_0 < k_1 < \cdots < k_r = k$, of positive integers. Set $k'_1 = k - k_i$.

Fix k, and let x, Y, and Δ be three indeterminates. Form the polynomial $P_k(x, Y, \Delta)$ given by

$$\sum_{\{k_1,\ldots,k_{r-1}\}} x^{k-r} \prod_{i=1}^{r} (k'_i + Y + \Delta(k_i - k_{i-1})) \left(\prod_{i=1}^{r-1} k_i k'_1 \right)^{-1} .$$

In the summation r is not fixed, so that the sum runs over all unordered partitions of k. The polynomial is of degree k in Y, and the coefficient of Y^k is $((k-1)!)^{-2}$.

It can be factored explicitly. For this it is convenient to write

$$\Delta = h_{p,q}(m) = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}.$$

Observe that if $m \neq 0, -1$ then, given Δ, this equation can always be solved for p and q. Set

$$\begin{aligned} Y_s(m) &= (((1-k)^2 - (p-q+s)^2)m^2 + 2((1-k) - (p-q+s)p)m + 1 - p^2)/4m(m+1) \\ &= h_{1,k}(m) - h_{p,q-s}(m), \\ Y'_r(m) &= (((k-1)^2 - (p-r-q)^2)m^2 + 2((k-1)k - (p-r-q)(p-r))m \\ &\quad + k^2 - (p-r)^2)/4m(m+1) \\ &= h_{k,1}(m) - h_{p-r,q}(m). \end{aligned}$$