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Analyticity of Scattering for the φ^4 Theory

John C. Baez[1] and Zheng-Fang Zhou[2]

[1] University of California, Riverside, CA 92521, USA

[2] Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Abstract. We consider scattering for the equation $(\Box + m^2)\varphi + \lambda\varphi^3 = 0$ on four-dimensional Minkowski space. For $m > 0$, one-to-one and onto wave operators $W_\lambda^{\pm}: \mathbf{H} \to \mathbf{H}$ are known to exist for all $\lambda \geqq 0$, where $\mathbf{H}$ denotes the Hilbert space of finite-energy Cauchy data. We prove that the maps $(\lambda, u) \mapsto W_\lambda^{\pm}(u)$ and $(\lambda, u) \mapsto (W_\lambda^{\pm})^{-1}(u)$ are continuous from $[0, \infty) \times \mathbf{H}$ to $\mathbf{H}$, and extend to real-analytic functions from an open neighborhood of $\{0\} \times \mathbf{H} \cup \mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbf{H}$ to the Hilbert space $\mathbf{H}_{-1}$ of Cauchy data with Poincaré-invariant norm. For $m = 0$, wave operators $W_\lambda^{\pm}$ are known to exist as diffeomorphisms of $\mathbf{H}$ for all $\lambda \geqq 0$, where here $\mathbf{H}$ denotes the Hilbert space of finite Einstein energy Cauchy data. In this case we prove that the maps $(\lambda, u) \mapsto W_\lambda^{\pm}(u)$ and $(\lambda, u) \mapsto (W_\lambda^{\pm})^{-1}(u)$ extend to real-analytic functions from a neighborhood of $[0, \infty) \times \mathbf{H} \subset \mathbb{R} \times \mathbf{H}$ to $\mathbf{H}$.

1. Introduction

The classical φ^4 theory is the Poincaré-invariant nonlinear wave equation:

$$(\Box + m^2)\varphi + \lambda\varphi^3 = 0, \quad m, \lambda \geqq 0,$$

where $\Box$ denotes the D'Alembertian on Minkowski space, $\mathbf{M}_0 \cong \mathbb{R}^4$, and φ is a real-valued function on $\mathbf{M}_0$. Its main interest is as a simple classical analogue of the equations describing interacting relativistic quantum fields, which in four dimensions have so far resisted attempts at a rigorous formulation. The possibility of the existence of wave and scattering operators for this equation as transformations of the Hilbert space $\mathbf{H}$ of finite-energy solutions of the free equation ($\lambda = 0$) was suggested by Segal, who first published results in this direction in 1966 [10]. The problem inspired a large amount of research, most focusing on the massive case ($m > 0$). In 1978 Strauss [12] proved for this case the existence of wave operators $W^{\pm}: \mathbf{H} \to \mathbf{H}$ such that

$$\lim_{t \to \pm\infty} \| U(t)u - V(t) W^{\pm} u \| = 0$$

for each $u \in \mathbf{H}$, where $U(t)$ is the unitary group on $\mathbf{H}$ corresponding to time evolution