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Scattering Theory in the Energy Space for a Class of Non-Linear Wave Equations

J. Ginibre[1] and G. Velo[2]

[1] Laboratoire de Physique Théorique et Hautes Energies,* Université de Paris XI, Bâtiment 211, F-91405 Orsay, France
[2] Dipartimento di Fisica, Università di Bologna and INFN, Sezione di Bologna, Italy

Abstract. We study the asymptotic behaviour in time of the solutions and the theory of scattering in the energy space for the non-linear wave equation

$$\square\varphi + f(\varphi) = 0$$

in $\mathbb{R}^n$, $n \geqq 3$. We prove the existence of the wave operators, asymptotic completeness for small initial data and, for $n \geqq 4$, asymptotic completeness for arbitrarily large data. The assumptions on f cover the case where f behaves slightly better than a single power $p = 1 + 4/(n-2)$, both near zero and at infinity (see (1.5), (1.6) and (1.8)).

1. Introduction

A large amount of work has been devoted to the theory of scattering for the non-linear wave (NLW) equation (or non-linear massless Klein-Gordon equation)

$$\square\varphi \equiv \ddot{\varphi} - \Delta\varphi = -f(\varphi), \tag{1.1}$$

where φ is a complex valued function defined in space time $\mathbb{R}^{n+1}$, the upper dot denotes the time derivative, Δ is the Laplace operator in $\mathbb{R}^n$ and f is a non-linear complex valued function, a typical form of which is

$$f(\varphi) = \lambda\varphi|\varphi|^{p-1} \tag{1.2}$$

with $1 \leqq p < \infty$. We refer to a previous paper [12] for a more detailed introduction and a comprehensive bibliography. It is known [9, 10] that the Cauchy problem for the equation (1.1) with initial data $(\varphi(t_0), \dot{\varphi}(t_0)) = (\varphi_0, \psi_0)$ at time t_0 in the space $H^1 \oplus L^2$ has a unique solution $(\varphi, \dot{\varphi}) \in \mathscr{C}(\mathbb{R}, H^1 \oplus L^2)$ under assumptions on f which reduce to $\lambda \geqq 0$ and to

$$0 \leqq p - 1 < \begin{cases} 4/(n-2) & \text{if} \quad n \geqq 3 \\ \infty & \text{if} \quad n \leqq 2 \end{cases} \tag{1.3}$$

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* Laboratoire associé au Centre National de la Recherche Scientifique