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A Note on the Diffusion of Directed Polymers in a Random Environment

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Abstract. A simple martingale argument is presented which proves that directed polymers in random environments satisfy a central limit theorem for $d \geqq 3$ and if the disorder is small enough. This simplifies and extends an approach by J. Imbrie and T. Spencer.

1. Introduction

In a recent paper, Imbrie and Spencer [1] considered the following model of a random walk in a random environment. Let $\xi(t)$, $t \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be an ordinary symmetric random walk on $\mathbf{Z}^d$ starting in 0 and let $h(t,x)$, $t \in \mathbb{N}$, $x \in \mathbf{Z}^d$, be i.i.d. random variables which are $+$ or -1 with probability 1/2 and also independent of ξ. We denote by $\langle \ \rangle$ the expectation with respect to ξ and by $E(\,.\,)$ the expectation with respect to the h-variables. Let $0 < \varepsilon < 1$ be fixed and for $T \in \mathbb{N}$,

$$\kappa(T) = \prod_{j=1}^{T} (1 + \varepsilon h(j, \xi(j))) \ .$$

Imbrie and Spencer proved the following result by a rather elaborate expansion technique:

Theorem 1. *If* $\varepsilon > 0$ *is small enough and* $d \geqq 3$, *then*

$$\lim_{T \to \infty} \langle |\xi(T)|^2 \kappa(T) \rangle / T \langle \kappa(T) \rangle = 1 \quad \textit{almost surely}$$

(here $|\ |$ *is the Euclidean norm).*

We give here a very simple proof based on martingale limit theorems. The result in [1] is somewhat stronger and includes also a convergence rate. Such rates can also be obtained by the method presented here. An inspection of the proof reveals that the convergence rate is $O(T^{-\delta})$ almost surely for $\delta < (d-2)/4$. Theorem 1 is a special case of a more general result which includes the central limit theorem which seems to be new. Let $\xi_1(T), \ldots, \xi_d(T)$ be the components of the random walk.