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A Remark on Smoothing of Magnetic Schrödinger Semigroups

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Abstract. We prove that, under mild regularity conditions, the magnetic Schrödinger semigroup generated by $H_0(\mathbf{a}) = \frac{1}{2}\sum_j (i\partial_j - a_j)^2$ has its range inside the bounded continuous functions. We also give a counterexample for the general case.

1. Definition. Let $a \in L^2_{\text{loc}}(\mathbb{R}^\nu, \mathbb{R}^\nu)$ such that $\operatorname{div} a$ (distributional divergence) is in $L^1_{\text{loc}}(\mathbb{R}^\nu)$. The magnetic Schrödinger semigroup is defined by the Feynman-Kac-Itô formula (see [1, Sects. 14–16] for an early review)

$$(f, \exp(-tH_0(\mathbf{a}))g) = \int_\Omega \exp(F(\omega, t))\overline{f(\omega(0))}g(\omega(t))d\mu_0(\omega),$$

where

$$F(\omega, t) = -i \int_{s=0}^{t} a(\omega(s)) \cdot d\omega - (i/2) \int_{s=0}^{t} (\operatorname{div} a)(\omega(s))ds.$$

Here μ_0 is full Wiener measure (the product of Wiener space with Lebesgue measure on the starting points) and the stochastic integral is taken in the sense of Itô.

2. Gauge Invariance. If a is increased by a gradient $\nabla\lambda$, the semigroup and its generator are transformed isometrically via multiplication by $\exp(i\lambda)$. It follows that the spectrum does not change. More precisely, we have

2.1. Theorem [2, p. 168]. *Let $a, b \in L^2_{\text{loc}}(\mathbb{R}^\nu, \mathbb{R}^\nu)$ and suppose that* $\operatorname{curl} a = \operatorname{curl} b$. *Then $H_0(a)$ is closable in L^2 if and only if $H_0(b)$ is. Moreover,*

$$\exp(i\lambda)H_0(a)\exp(-i\lambda) = H_0(b). \quad \square$$

3. Smoothing. We now consider $a \in L^2_{\text{loc}}$ such that $a^2 \in K_\nu^{\text{loc}}$ (see e.g. [3, p. 453] for a definition and a review of properties).

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