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The Existence of the Thermodynamic Limit in Coulomb-like Systems

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Abstract. A Coulomb-like system is a system in which the usual $1/r$ Coulomb potential has been replaced by a potential that goes as $1/r^\alpha$ for α near one. Such potentials do not have the mean value property, which forms the basis of the Lebowitz–Lieb argument to control the long range Coulomb interaction [5], so whether or not such systems actually exhibit thermodynamic behavior is an interesting question. In this paper I generalize the proof of the limit for a crystal with Coulomb potential given by Charles Fefferman [1] to cover these Coulomb-like potentials.

1. Introduction

The statistical mechanics of systems of electrons and protons interacting via the Coulomb interaction has long been a subject of interest. One would like to know a little more about systems with a pair interaction that goes as some power $\alpha \neq 1$ of the inverse distance simply to answer the question "In what sense is the Coulomb potential special?" In this paper I hope to demonstrate that the Coulomb potential ($\alpha = 1$) is not special, at least in the sense that as one moves away from $\alpha = 1$ nothing catastrophic happens and the resulting systems still have a thermodynamic limit.

First, the basic set-up. The states of our system of N_1 electrons and N_2 protons are taken to be the eigenfunctions of the quantum mechanical Hamiltonian

$$H_{N,B_R} = \sum_{j=1}^{N_1} -\kappa_1 \Delta_{x_j} + \sum_{k=1}^{N_2} -\kappa_2 \Delta_{y_k} + \frac{1}{2}\sum_{j\neq 1} \frac{1}{|x_j - x_l|^\alpha} + \frac{1}{2}\sum_{k\neq m} \frac{1}{|y_k - y_m|^\alpha} - \sum_{j,k} \frac{1}{|x_j - y_k|^\alpha}$$

with Dirichlet boundary conditions on some large ball B_R. The eigenfunctions $\psi(x_1, \ldots, x_{N1}, y_1, \ldots, y_{N2})$ are assumed to all be L^2 functions that are separately antisymmetric in the x and y variables. In the following the appropriate L^2 space will be denote $L^2((B_R)^{N1+N2})$ or $L^2_N(B_R)$. The operator we have written down is self-adjoint for $0 < \alpha < 2$ (see, e.g. Reed & Simon [6]), so we will consider that range of α in the following.

For convenience we will consider only the grand canonical ensemble: a