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The Hannay Angles: Geometry, Adiabaticity, and an Example

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Abstract. The Hannay angles were introduced by Hannay as a means of measuring a holonomy effect in classical mechanics closely corresponding to the Berry phase in quantum mechanics. Using parameter-dependent momentum mappings we show that the Hannay angles are the holonomy of a natural connection. We generalize this effect to non-Abelian group actions and discuss non-integrable Hamiltonian systems. We prove an averaging theorem for phase space functions in the case of general multi-frequency dynamical systems which allows us to establish the almost adiabatic invariance of the Hannay angles. We conclude by giving an application to celestial mechanics.

1. Introduction

Consider a classical system whose Hamiltonian $H(r)$ depends smoothly on a set of time-dependent parameters r. Hannay [21] and Berry [8] have shown that, under a closed adiabatic loop in the space of classically integrable Hamiltonians, the angle variables pick up extra angles, the *Hannay angles*, in addition to the time integral of the instantaneous frequencies. (Here the term *adiabatic* means that the time dependence of the parameters is assumed to be slow.) Hannay explains these angles by the fact that the action-angle coordinates $(J, \varphi) \in \mathbb{R}^n \times \mathbb{T}^n$ are parameter-dependent so that the canonical transformation to these coordinates produces an additional term in the Hamiltonian. More explicitly, let $H(p, q, r)$ be an integrable Hamiltonian for all fixed values of the parameters. When the parameters $r = r(\varepsilon t)$ change in time, $r(s+T) = r(s)$, dynamics is given by the time-dependent Hamiltonian

$$h = h_0(J, \varepsilon t) + \varepsilon h_1(J, \varphi, \varepsilon t), \tag{1}$$

where h_0 is just the original Hamiltonian expressed in action variables, whereas εh_1 arises from the time-derivative of the generating function of the canonical

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