Commun. Math. Phys. 122, 575–596 (1989)

Communications in Mathematical Physics

The Two-Dimensional $O(N)$ Nonlinear σ-Model: Renormalisation and Effective Actions

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Abstract. We establish the existence of the Wilson Renormalised trajectory of the $O(N)$ sigma model in perturbation theory in the "effective charge." This yields a proof of perturbative renormalisability, and is also relevant in the "small-field" analysis of the rigorous renormalisation group construction of the continuum theory.

1. Introduction

The two-dimensional nonlinear $O(N)$ sigma model, with $N \geqq 3$, is perturbatively renormalisable and asymptotically free [10, 1, 2]. In this work we study the model from the Wilson Renormalisation Group (RG) viewpoint and show the existence of the renormalised trajectory in perturbation theory in the "effective charge." This yields in particular a proof of perturbative renormalisability and the expansion in the small-field region that would be part of a rigorous RG construction of the model. The approach is similar to that of J. Polchinski [9], where the $\lambda\phi^4$ theory in four dimensions is treated. There are, however, marked differences and surprising simplifications. We *do not* break the symmetry by applying a magnetic field, and the analysis is therefore not around the Gaussian fixed point. Only two marginal directions are involved, and these can be isolated very cleanly, yielding a surprisingly pleasant proof of renormalisability.

The model is the quantum field theory of maps $\mathbb{R}^2 \to S^{N-1}$. With a lattice cut-off a, the theory is defined by a $\mathbb{R}^N$-valued field ϕ on $a\mathbb{Z}^2 \subset \mathbb{R}^2$ with the constraint $\phi^2 = 1/Z_0(a)$, and the bare action

$$S(a) = \frac{Z_0(a)}{2g_0^2(a)} \sum_{\substack{x,y \in \mathbb{Z}^2 \\ x,y \text{ n.nbs}}} \frac{[\phi(x) - \phi(y)]^2}{2}.$$

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*** Research funded in part by NSF grant DMS 8601978