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Monodromy in the Quantum Spherical Pendulum

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Abstract. In this article we show that monodromy in the quantum spherical pendulum can be interpreted as a Maslov effect: i.e. as multi-valuedness of a certain generating function of the quantum energy levels.

1.

A classical dynamical system of n degrees of freedom is completely integrable if it admits n independent integrals of motion, $H_1, \ldots, H_n$, with the property $\{H_i, H_j\} = 0$. The standard method for integrating such a system is by means of action-angle variables. The action variables, $I_1, \ldots, I_n$ can, in principle, be computed by simple quadratures and the angle variables, $\theta_1, \ldots, \theta_n$ determined by the Darboux relation $\Omega = \sum dI_i \wedge d\theta_i$; and, in terms of them, the system takes the transparently simple form:

$$\dot{\theta}_i = \frac{\partial H}{\partial I_i}(I_1, \ldots, I_n), \quad \dot{I}_i = 0. \tag{1.1}$$

In [D] Duistermaat raised the question of whether Eq. (1.1), appropriately interpreted, make sense globally on the entire phase space of the system in question. He showed this to be the case iff certain topological invariants are zero. The most important of these is an invariant, called monodromy, which we will describe in Sect. 3. One integrable system for which it is non-zero is the spherical pendulum.

Recently Cushman and Duistermaat [C.D] detected monodromy effects in the quasi-classical behavior of the spherical pendulum. For the spherical pendulum the quasi-classical values of the energy and axial angular momentum form a set in the plane which looks like a twisted lattice. They show, however, that because of monodromy, it is impossible to construct an injective map of this set into the standard lattice, $\hbar\mathbb{Z}^2$, which scales properly as $\hbar \to 0$.

The question we will investigate below is: Can monodromy be detected in the *quantum* behavior of the spherical pendulum? More explicitly, consider the Schrödinger operator on S^2

$$-\hbar^2 \Delta + V, \tag{1.2}$$