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S^1 Actions and Elliptic Genera*

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Abstract. A proof is given of Witten's conjectures for the rigidity of the index of the Dirac-Ramond operator on the loop space of a spin manifold which admits an S^1 symmetry.

1. Introduction

When M is a connected, compact, oriented, even dimensional, spin Riemannian manifold, one can define the Dirac operator, ∂, to act on the space of smooth sections of the bundle of complex spinors, $S(T^*M) \to M$. The index of this operator can be defined by using Clifford multiplication on $S(T^*M)$ by $(i)^{n(n+1)/2} \cdot \omega$, with ω being the image in the Clifford algebra of the volume form on M and with $n = \dim(M)$. This defines a covariantly constant involution, γ, of $S(T^*M)$. As an involution of $C^\infty(S(T^*M))$, γ anti-commutes with the Dirac operator. Then,

$$\operatorname{Ind}(\partial, \gamma) \equiv \dim(\ker(\partial|_{\ker(\gamma-1)})) - \dim(\ker(\partial|_{\ker(\gamma+1)})). \tag{1.1}$$

Now, suppose that M admits an isometric action of S^1. Here, the index of ∂ has a refinement which is the S^1 equivariant index. That is, use the S^1 action to decompose $C^\infty(S(T^*M)) = \oplus_k C^\infty(S(T^*M), k)$ where the double cover of S^1 acts on $C^\infty(S(T^*M), k)$ as multiplication by λ^k; $\lambda \in S^1$. As ∂ and γ commute with the S^1 action, they both preserve $C^\infty(S(T^*M), k)$ and with this understood, the S^1-equivariant index of ∂ is, by definition, the set of integers, $\{\operatorname{Ind}(\partial, \gamma, k)\}$, which is obtained by replacing $\ker(\gamma \pm 1) \cap C^\infty(S(T^*M))$ in Eq. (1.1) with $\ker(\gamma \pm 1) \cap C^\infty(S(T^*M), k)$.

The S^1-equivariant index can be generalized in the usual way by twisting the dirac operator with a vector bundle over M. Thus, when $V \to M$ is a complex vector bundle, one can define the index of the Dirac operator on $S(T^*M) \otimes V$, $\operatorname{Ind}(\partial, V, \gamma)$, by replacing $\ker(\gamma \pm 1) \subset C^\infty(S(T^*M))$ with $\ker(\gamma \pm 1) \subset C^\infty(S(T^*M) \otimes V)$. And, if a finite cover of the S^1 action on M has a lift to V, one can consider the S^1 equivariant

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