Commun. Math. Phys. 122, 321–354 (1989)

Communications in
Mathematical
Physics

The Kowalewski Top 99 Years Later: A Lax Pair, Generalizations and Explicit Solutions

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Abstract. A "natural" Lax pair for the Kowalewski top is derived by using a general group-theoretic approach. This gives a new insight into the algebraic geometry of the top and leads to its complete solution via finite-band integration theory.

Introduction

In her celebrated paper [1] published in 1889 Kowalewski found a new and highly nontrivial integrable case of the motion of a heavy rigid body about a fixed point, completing the list of integrable tops. Two previously known integrable cases are relatively simple and had been solved already in the XVIII[th] century. These are Euler's top in which the stationary point coincides with the center of mass, and Lagrange's top which is axially symmetric. The third case discovered by Kowalewski is rather bizarre: the moments of inertia have a fixed ratio 2 : 2 : 1, and the center of mass lies in the equatorial plane of the top. This case was detected by requiring that the general solution be given by meromorphic functions of the complex time variable. Unlike most other integrable systems of mechanics known in the XIX[th] century, the Kowalewski top cannot be solved by separation of variables. To integrate it, Kowalewski used an ingeneous change of variables which reduced the problem to hyperelliptic quadratures. However, the inverse change of variables leads to highly complicated expressions in terms of hyperelliptic theta functions which seem completely unmanageable (explicit formulae for the "physical" variables of the top were derived by Kötter [2]).

The new powerful method of finite-band integration created by Novikov, Dubrovin, Matveev, Its, Krichever and others (see [3] for a review) has led to a revival of interest in integrable problems of mechanics. Lax pairs have been found for many classical integrable systems, in particular, for various integrable cases of motion of a rigid body and for their multi-dimensional analogs. The Kowalewski top, however, has long remained a remarkable exception which baffled numerous attempts to build an adequate Lax pair for it. The only Lax pair known to the authors before 1987 was proposed by Perelomov [4]; it did not contain a spectral parameter and so was of no help in solving the equations of motion.