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Differential Calculus on Compact Matrix Pseudogroups (Quantum Groups)

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Abstract. The paper deals with non-commutative differential geometry. The general theory of differential calculus on quantum groups is developed. Bicovariant bimodules as objects analogous to tensor bundles over Lie groups are studied. Tensor algebra and external algebra constructions are described. It is shown that any bicovariant first order differential calculus admits a natural lifting to the external algebra, so the external derivative of higher order differential forms is well defined and obeys the usual properties. The proper form of the Cartan Maurer formula is found. The vector space dual to the space of left-invariant differential forms is endowed with a bilinear operation playing the role of the Lie bracket (commutator). Generalized antisymmetry relation and Jacobi identity are proved.

0. Introduction

It is difficult to overestimate the importance of the methods of differential geometry in the theory of Lie groups. Such fundamental notions as tangent Lie algebra and infinitesimal representation are provided by these methods. Infinitesimal representations in turn play an important role in applications of Lie groups in physics: Infinitesimal generators are related to the most important observables such as energy-momentum, angular momentum, and internal quantum numbers.

On the other hand, for differential geometry the Lie group theory is much more than just one of the fields of application. It is sufficient to dip into any contemporary handbook of differential geometry to see that Lie groups lay behind such basic notions as principal and associated fibre bundles. In fact, the development of differential geometry in the past 30 years can be considered as a far reaching implementation of the Erlangen program of F. Klein.

We believe that a similar interplay between differential geometry and group theory will be very fruitful also in the theory of non-commutative spaces. In this case instead of Lie groups one has to consider more general objects: non-