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The Coexistence Problem for the Discrete Mathieu Operator

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Abstract. We solve the coexistence problem for the periodic discrete Mathieu operator in all parametric cases. The main tool in the proof will be Bezout's theorem for projective plane curves. As an additional result we obtain the gap opening and gap growth powers for this operator.

0. Introduction and Main Results

Define for $b:\mathbf{Z}\to\mathbf{R}$ the linear recursion operator $H_b:\mathbf{C}^{\mathbf{Z}}\to\mathbf{C}^{\mathbf{Z}}$ by

$$(H_b g)(n) = g(n+1) + b_n g(n) + g(n-1).$$

There exists an actual interest in the cases where the potential b is almost periodic, in particular in the discrete Mathieu operator (with real parameters A, α, v) which one obtains by taking $b = Ab^{(\alpha,v)}$ with $b_n^{(\alpha,v)} = 2\cos(2\pi n\alpha - v)$. Thus explicitly

$$(H_{Ab^{(\alpha,v)}} g)(n) = g(n+1) + 2A\cos(2\pi n\alpha - v)g(n) + g(n-1).$$

This operator, a discrete version of Mathieu's periodic differential operator, is interesting from the mathematical as well from the physical point of view [1-7, 9, 12, 16, 18, 19, 21]. Here we shall occupy ourselves only with the periodic potential case, i.e., where α is a rational number p/q (always written in its canonical form). The following theorem completely solves the coexistence problem for the periodic discrete Mathieu operator.

Theorem 1. *For all*[1] $A\in\mathbf{R}^*$, $p/q\in\mathbf{Q}$, $v\in\mathbf{R}$ *all the* $q-1$ *gaps of the periodic discrete Mathieu operator* $H_{Ab^{(p/q,v)}}$ *are non-degenerate, with the exception of the middle gaps (which are* $\{0\}$*) in the cases* $v\in(2\pi/q)\mathbf{Z}$, $q\equiv 0 \pmod 4$ *and the cases* $v\in(\pi/q)+(2\pi/q)\mathbf{Z}$, $q\equiv 2 \pmod 4$. ◇

The coexistence problem for Mathieu's differential operator has been solved by Ince [14]. Partial results for Theorem 1 had already been obtained in [3, 6]. The

[1] For a subset X of C we denote $X\backslash\{0\}$ by X^*