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Density and Uniqueness in Percolation

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Abstract. Two results on site percolation on the d-dimensional lattice, $d \geqq 1$ arbitrary, are presented. In the first theorem, we show that for stationary underlying probability measures, each infinite cluster has a well-defined density with probability one. The second theorem states that if in addition, the probability measure satisfies the finite energy condition of Newman and Schulman, then there can be at most one infinite cluster with probability one. The simple arguments extend to a broad class of finite-dimensional models, including bond percolation and regular lattices.

Our notation is as follows. Let $\mathbf{Z^d}$ be the d-dimensional lattice, with $d \geqq 1$. Finite d-dimensional boxes in $\mathbf{Z^d}$ whose sides are parallel to the coordinate directions are called *rectangles*. The number of points in a finite set F is denoted by $\#(F)$. We set

$$X = \{0, 1\}^{\mathbf{Z^d}},$$

and we let μ denote a probability measure on X which is *stationary*, i.e. invariant under translation by each element of $\mathbf{Z^d}$. For each $x \in X$, the connected components of the nearest neighbor graph whose set of vertices is

$$\{z \in \mathbf{Z^d} \colon x(z) = \mathbf{1}\}$$

are called *clusters*.

Definition. A subset S of $\mathbf{Z^d}$ has *density* α if for each sequence of rectangles $R_1 \subseteqq R_2 \subseteqq \dots$ with $\bigcup_{n \geqq 1} R_n = \mathbf{Z^d}$, the limit

$$\lim_{n \to \infty} \frac{\#(S \cap R_n)}{\#(R_n)}$$

exists and is equal to α. If S does not have *density* [α for any α], then S is *rough*.

Theorem 1. *For μ-almost every $x \in X$, each cluster of x has density.*