Commun. Math. Phys. 121, 351–399 (1989)

Communications in
Mathematical
Physics

Quantum Field Theory and the Jones Polynomial *

Edward Witten **

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Abstract. It is shown that $2+1$ dimensional quantum Yang-Mills theory, with an action consisting purely of the Chern-Simons term, is exactly soluble and gives a natural framework for understanding the Jones polynomial of knot theory in three dimensional terms. In this version, the Jones polynomial can be generalized from S^3 to arbitrary three manifolds, giving invariants of three manifolds that are computable from a surgery presentation. These results shed a surprising new light on conformal field theory in $1+1$ dimensions.

In a lecture at the Hermann Weyl Symposium last year [1], Michael Atiyah proposed two problems for quantum field theorists. The first problem was to give a physical interpretation to Donaldson theory. The second problem was to find an intrinsically three dimensional definition of the Jones polynomial of knot theory. These two problems might roughly be described as follows.

Donaldson theory is a key to understanding geometry in four dimensions. Four is the physical dimension at least macroscopically, so one may take a slight liberty and say that Donaldson theory is a key to understanding the geometry of space-time. Geometers have long known that (via de Rham theory) the self-dual and anti-self-dual Maxwell equations are related to natural topological invariants of a four manifold, namely the second homology group and its intersection form. For a simply connected four manifold, these are essentially the only classical invariants, but they leave many basic questions out of reach. Donaldson's great insight [2] was to realize that moduli spaces of solutions of the self-dual Yang-Mills equations can be powerful tools for addressing these questions.

Donaldson theory has always been an intrinsically four dimensional theory, and it has always been clear that it was connected with mathematical physics at least at the level of classical nonlinear equations. The puzzle about Donaldson theory was whether this theory was tied to more central ideas in physics, whether it could be interpreted in terms of quantum field theory. The most important

* An expanded version of a lecture at the IAMP Congress, Swansea, July, 1988

** Research supported in part by NSF Grant No. 86-20266, and NSF Waterman Grant 88-17521