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Eigenvalue Branches of the Schrödinger Operator $H-\lambda W$ in a Gap of $\sigma(H)$

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Abstract. The authors study the eigenvalue branches of the Schrödinger operator $H-\lambda W$ in a gap of $\sigma(H)$. In particular, they consider questions of asymptotic distribution of eigenvalues and bounds on the number of branches. They also address the completeness problem.

Introduction

Let $V(x)$, $W(x)$ be real bounded functions on $\mathbf{R}^\nu$ satisfing

(a) $$V(x) \geqq 1,$$

(b) $$\lim_{|x|\to\infty} W(x) = 0.$$

Let H denote the self-adjoint operator $-\Delta + V$ on $L^2(\mathbf{R}^\nu)$.

This paper is devoted to the study of three questions concerning the eigenvalue branches of the family of Schrödinger operators $H \pm \lambda W$, in a gap of $\sigma(H)$:

(1) For $W \geqq 0$ we consider the asymptotics of the number of branches which cross an energy E in the gap and which emerge from below. To be more precise, we compute the number of branches of $H+\mu W$ which cross the level $E \in \mathbf{R}-\sigma(H)$ for $0<\mu<\lambda$, as $\lambda\to\infty$.

(2) When $W \geqq 0$ and $\operatorname{supp} W$ is contained in B_R, the ball of radius R, we prove a semi-classical phase-space type bound on the number of eigenvalue branches of the family $H+\lambda W$, $\lambda>0$, which cross a given level E in the gap. In particular, we show that the total number of such branches is finite and is bounded by the volume of the ball B_R,

$$\#\{\text{branches } E_j(\lambda) \text{ which cross } E\} \leqq C_0 R^\nu,$$

where C_0 is independent of $W \in L^\infty(B_R)$, $W \geqq 0$, so long as $\operatorname{supp} W \subset B_R$.

(3) We address the "completeness problem" (cf. Deift and Hempel [DH]) for W which change sign; i.e., for *each* E in the gap, does there exist a $\lambda>0$ so that $E \in \sigma(H-\lambda W)$?