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Integro-Differential Non-Linear Equations and Continual Lie Algebras

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Abstract. The integrability problem of integro-differential equations with, generally speaking, singular kernels is discussed after an example of new continual analogs of the two-dimensional Toda lattices. These equations are associated with new infinite-dimensional Lie algebras via zero curvature type representation. The structural constants of these algebras are distributions. A formal solution of the Goursát problem is obtained. For the case with the kernel of the integral operator being $\delta_{\pm}$-distribution an explicit expression in quadratures for the solutions is given.

1. Introduction

Recently an interest in the integrability problem for nonlinear integro-differential equations revived, mainly due to physicist-theoreticians and applied mathematicians. There would be no harm to note that this problem has been in a stagnant state for a long period. The renaissance in question is motivated, in particular, by the problems of fluid flow dynamics and plasma physics. Probably, it is not alien also to the gauge theories in particle physics. In this, the integro-differential equations such as Benjamin-Ono or sine-Hilbert are usually studied. These equations are derived from the wave or evolution differential equations by introducing in one of the terms an integral operator with the Cauchy or Hilbert kernel over a spatial argument (see, for example [1]). However, at present there are not any sufficiently general and self-consistent constructions of nonlinear integro-differential equations and the integrability criteria for them.

Thus, it seems reasonable for this goal to try to generalize an algebraic approach which proved itself in a good light in the problem of constructing and explicit investigation of the integrable differential systems [2]. Here we take as the starting point in the spirit of the Volterra method [3] for transition from finite to infinite number of variables, a continual extension of the multicomponent differential systems. In the present paper the two-dimensional generalized Toda lattice is used as an example of such an initial differential system. Its continual