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Stability of Semiclassical Bound States of Nonlinear Schrödinger Equations with Potentials

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Abstract. In this paper, we study the Lyapunov stabilities of some "semiclassical" bound states of the (nonhomogeneous) nonlinear Schrödinger equation,

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2}\Delta\psi+V\psi-|\psi|^{p-1}\psi,\quad 1\leqq p<1+\frac{4}{n}.$$

We prove that among those bound states, those which are "concentrated" near local minima (respectively maxima) of the potential V are stable (respectively unstable). We also prove that those bound states are positive if $\hbar>0$ is sufficiently small.

1. Introduction

In [W.a] and [FW], the following nonlinear Schrödinger equation (abbreviated as NLS) on $\mathbf{R}$

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{1}{2}\hbar^2\frac{d^2\psi}{dx^2}+V\psi-|\psi|^2\psi \tag{1}$$

was proposed to study to stabilize linear modes concentrated near local minima for sufficiently small $\hbar>0$ for potentials bounded below. Unlike the linear case, Floer and Weinstein proved the existence of solutions of (1) for sufficiently small $\hbar>0$, which is localized near each nondegenerate critical point of V for all time. We call these solutions "semiclassical solutions." In [O3], the present author generalized the existence result for arbitrary potentials with mild restrictions on the oscillations of V at infinity. Let us briefly summarize the existence result in [FW] and [O3]: If we rescale time and space by $t\to\hbar s$ and $x\to\hbar y$, then rewriting s by t, Eq. (1) becomes

$$i\frac{\partial\psi}{\partial t}=-\frac{1}{2}\frac{\partial^2\psi}{\partial y^2}+V_\hbar\psi-|\psi|^2\psi, \tag{2}$$

where $V_\hbar(y)=V(\hbar y)$. Without loss of generalities, we assume that 0 is the critical point we are considering and that $V(0)=0$. Then as $\hbar\to 0$, $V_\hbar\to 0$ uniformly over