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Pseudo-Differential Projections and the Topology of Certain Spaces of Elliptic Boundary Value Problems

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Abstract. We calculate the homotopy groups of the space of elliptic boundary value problems for an elliptic differential operator A of a first order and of the space of elliptic self-adjoint boundary value problems when A is a formally self-adjoint. In particular we show that the spectral flow of an S^1 family of self-adjoint elliptic boundary value problems is well defined. This provides some information on spectral properties along the lines of the Vafa–Witten approach to spectral inequalities.

1. The Notion of an Elliptic Boundary Value Problem

Let X be a smooth compact manifold with boundary Y. Let E and F be smooth complex vector bundles over X. For simplicity, we consider $A: C^\infty(X;E) \to C^\infty(X;F)$ an elliptic differential operator over X of first order. We discuss the results for a larger class of operators at the end of the paper. Now let us fix a Riemannian metric on X and Hermitian structures on E and F. Then in a fixed collar neighbourhood $N \cong I \times Y$ of $Y \cong 0 \times Y$ the operator A takes the form

$$A(t,y) = G(t,y)(\partial_t + B_t(y)), \tag{1}$$

where $G(t,y): E|_Y \to F|_Y$ is a bundle isomorphism for fixed t; ∂_t is the normal derivative; and $B_t: C^\infty(Y;E|_Y) \to C^\infty(Y;E|_Y)$ is an elliptic differential operator of first order on Y such that the principal symbol $b_t(y,\zeta)$ of B_t has no purely imaginary eigenvalues. We will assume that $G(t,y)$ is unitary.

It is well known that the orthogonal (with respect to $L^2(Y;E|_Y)$) projection

$$P(A): C^\infty(Y;E|_Y) \to H(A)$$

of $C^\infty(Y;E|_Y)$ onto $H(A) = \{u|_Y \mid u \in C^\infty(X;E) \text{ and } Au = 0 \text{ in } X \setminus Y\}$ is a pseudo-differential operator of order zero. It is called the *Calderón projector* of A. At each point (y,ζ) of the cotangent sphere bundle SY the principal symbol of $P(A)$ is the orthogonal projection $p^+(y,\zeta): E_y \to E_y$ onto the direct sum $E^+_{y,\zeta}$ of the eigenspaces of the symbol $b_0(y,\zeta)$ corresponding to the eigenvalues with positive real part.

Now let V be a Hermitian bundle over Y and let $R: C^\infty(Y;E|_Y) \to C^\infty(Y;V)$ be