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Relating Kac–Moody, Virasoro and Krichever–Novikov Algebras

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Abstract. We demonstrate that the Kac–Moody and Virasoro-like algebras on Riemann surfaces of arbitrary genus with two punctures introduced by Krichever and Novikov are in two ways linearly related to Kac–Moody and Virasoro algebras on S^1. The two relations differ by a Bogoliubov transformation, and we discuss the connection with the operator formalism.

1. Introduction

Two-dimensional conformal field theories [1] have been considerably developed recently. In particular, they are relevant in the study of string multiloop amplitudes, which amount to the contribution of higher genus Riemann surfaces to partition functions and expectation values. The application of powerful mathematical results in algebraic geometry and in complex analysis on Riemann surfaces has led to a rather detailed understanding of the multiloop structure, especially in the operator formalism which uses punctured Riemann surfaces to describe scattering amplitudes [2, 3, 4].

On the other hand, the older algebraic approach consists of using the Kac–Moody and Virasoro algebras to describe the Hilbert space of a closed string. These algebras are then defined on S^1. They can be naturally extended to the Riemann sphere CP_1 with punctures at $z=0$ and $z=\infty$. This allows, for instance, the algebraic construction of the non-interacting string partition function.

Krichever and Novikov [5] introduced a natural extension of these algebras to the interacting string theory by formulating the algebras on a Riemann surface Σ of arbitrary genus g with two punctures at $P_{\pm}$. Note that for $g \geq 1$, these punctures cannot be moved to two specified points by conformal transformations as was the case for CP_1. The Kac–Moody algebra is defined as that of Lie algebra-valued meromorphic functions on Σ which are holomorphic outside $P_{\pm}$. Similarly, the Virasoro algebra is given by the algebra of meromorphic vector fields on Σ, holomorphic outside $P_{\pm}$. As in the operator formalism, one associates a set of local complex coordinates $z_{\pm}$ with the punctures $P_{\pm}$ that vanish at the puncture. As for the sphere, the radial parameter is related to the time parameter τ, such that P_- corresponds to $\tau=\infty$ and P_+ to $\tau=-\infty$. A coordinate-