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The Renormalization Flow, Spaces of Two-Dimensional Field Theories, and Connes' Geometry

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Abstract. We formulate a local renormalization flow using Connes' non-commutative geometry. This formulation allows a geometric description of the renormalization flow, and an intrinsic characterization of the operator product expansion. We define spaces for string theory, in terms of a ring of correlation functions and a renormalization flow on this ring, which are the analogues of the category of Riemannian manifolds with metric for general relativity. The beta function is related to a differential form of relative entropy between two renormalization flow trajectories.

1. The Problem

The problem of characterizing spaces of two-dimensional field theories originates in string theory, where one would like to understand what a background independent formulation of string theory might involve. However, since the deepest aspects of string theory involve two-dimensional critical phenomena, this problem is of interest in the context of statistical mechanics as well. The "classical" equation of motion of string theory, as we presently understand it, is the condition that the nonlinear sigma model describing string propagation in a particular background be conformally invariant, which is to say, describing behaviour at a (second or higher order) phase transition. An understanding of what spaces one might embed these critical models into, might be considered analogous to understanding that solutions to Einstein's equation are objects in the category of Riemannian manifolds with metric. Furthermore, given an action, or equivalently, in terms of the Feynman path integral, a measure on the set of objects of the category, such an embedding defines the dynamics and invariance structure of the theory. The *naïvest possible* embedding one can envisage in string theory is that with which we are concerned in this paper, the embedding of solutions of string theory into spaces of two-dimensional field

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