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Small Random Perturbations of Dynamical Systems: Exponential Loss of Memory of the Initial Condition

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Abstract. We consider Markov processes arising from small random perturbations of non-chaotic dynamical systems. Under rather general conditions we prove that, with large probability, the distance between two arbitrary paths starting close to a same attractor of the unperturbed system decreases exponentially fast in time. The case of paths starting in different basins of attraction is also considered as well as some applications to the analysis of the invariant measure and to elliptic problems with small parameter in front to the second derivatives. The proof is based on a multiscale analysis of the typical trajectories of the Markov process; this analysis is done using techniques involved in the proof of Anderson localization for disordered quantum systems.

Section 0. Introduction

This paper is concerned with the study of the dependence on the initial condition of some Markov processes X_t^ε which arise from small random perturbations of dynamical systems. These kind of processes arise more and more frequently in different areas of natural sciences like theoretical physics, statistical mechanics, chemistry, ergodic theory and ecology (see e.g. [1–5]) and their mathematical theory has been developed first by Ventzel and Freidlin in their basic work [6] and by Kiefer [7]. Following their ideas Gora analyzed the discrete case [8].

More precisely we will treat two different examples:

a) the discrete case, $t \in N$ in which the Markov chain X_t^ε is obtained by randomly composing two maps T and T^ε from a compact differentiable manifold M, with T^ε very close to the identity as $\varepsilon \to 0$.

b) the continuous case when the process X_t^ε is a diffusion process in R^n, solution of an ordinary Ito equation:

$$dX_t^\varepsilon = b(X_t^\varepsilon)dt + \varepsilon dw_t. \tag{0.1}$$

The precise hypotheses on T and T^ε are stated in Sects. 1 and 2. Basically we assume that the deterministic dynamical systems $x \to T(x)$ or $dx_t = b(x_t)dt$ have only a finite number of asymptotically stable fixed points or periodic orbits with