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Absence of Weak Local Rules for the Planar Quasicrystalline Tiling with the 8-fold Rotational Symmetry

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Abstract. Levitov's theory of local rules (the preceding paper of this issue) gives no prediction for the planar quasicrystalline tiling having 8-fold rotational symmetry. Absence of weak local rules for this particular tiling is proven.

A general theory of local rules for planar tilings has been developed by Levitov [1]. Below we study one of the simplest particular cases: a planar tiling with the 8-fold symmetry. This tiling can be obtained by projecting from the four-dimensional space. Matrix elements of the projector contain only one irrational number $\sqrt{2}$, which is a quadratic irrationality. Nevertheless, Levitov's Theorem 3 which states that quadratic irrationalities give rise to tilings with at least weak local rules, cannot be applied to our particular case. The theorem can be applied only to the tilings with the non-degenerate SI-conditions [1]. It is easy to show that in our case the SI-conditions are degenerate. Specific properties of the continued fraction $\sqrt{2} = \{1, 2, 2, 2, 2 \ldots\}$ along with the four-dimensionality of the total space and the 8-fold symmetry of the tiling enable one to prove the absence of weak local rules directly, without use of the general theory. Evidence for the absence of strong local rules for this particular tiling was given by Beenker [2]. Absence of strong local rules does not result in absence of weak rules, but absence of weak rules results in absence of strong rules. Thus, Beenker's statement is a consequence of the assertion proven below.

Definitions of tiles, tilings, quasicrystalline tilings, the lattice surface in total space, strong and weak local rules are given in the Levitov's paper [1]. We shall not give them once again. Define the particular tiling with the 8-fold symmetry. The tiling is quasicrystalline, i.e. it can be treated as a projection of the 2-D lattice surface confined in the standard tube onto some 2-D subspace $\mathbb{R}^2_{\sqrt{2}} \subset \mathbb{R}^4$ [1, 3]. We set this subspace by fixing the projections of unit basic vectors of $\mathbb{R}^4$ onto $\mathbb{R}^2_{\sqrt{2}}$:

$$\mathbf{e}'_1 = \frac{1}{2}\left(1; \frac{\sqrt{2}}{2}; 0; -\frac{\sqrt{2}}{2}\right), \qquad |\mathbf{e}'_j| = \frac{\sqrt{2}}{2}$$

$$\mathbf{e}'_2 = \frac{1}{2}\left(\frac{\sqrt{2}}{2}; 1; \frac{\sqrt{2}}{2}; 0\right),$$