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Local Rules for Quasicrystals

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Abstract. The relationship of local ordering and long-range order is studied for quasicrystalline tilings of plane and space. Two versions of the concept of local rules are introduced: strong and weak. Necessary conditions of the existence of strong local rules are found. They are mainly reduced to the constraints for irrational numbers related to incommensurabilities of the quasicrystals. For planar quasicrystals the quadratic irrationalities $a+b\sqrt{D}$ ($a, b \in \mathbb{Q}$, $D \in \mathbb{Z}$) play an important role. For three-dimensional quasicrystals not only quadratic but also cubic irrationalities $a+b\sqrt[3]{D}+c\sqrt[3]{D^2}$ ($a, b, c \in \mathbb{Q}$, $D \in \mathbb{Z}$) are allowed. The existence of weak local rules is established for almost all two-dimensional quasicrystals based on quadratic irrationalities and for the three-dimensional quasicrystal having icosahedral symmetry.

1. Introduction

Quasicrystals discovered by Schechtman, Blech, Gratias, and Cahn [1] are materials with the long-range order of a new type. They exhibit scattering properties of ideal crystals but have a point symmetry group incompatible with periodicity. Unusual diffraction properties of quasicrystals were explained with the help of periodic structures in high-dimensional spaces by Kalugin et al. [2]; Elser [3]; Duneau and Katz [4] (see also Mackay [5]; Levine and Steinhardt [6]; Kramer and Neri [7]). All the models of the atomic structure of quasicrystals discussed so far include some periodic structures in high-dimensional spaces (for the icosahedron symmetry group the dimension of the space is six and for the pentagon symmetry group it is five). The physical space is embedded into the high-dimensional space as an incommensurate subspace and the positions of atoms in it are found by projection. In another version the positions of atoms are given by the common points of the periodic structure and the physical subspace. The models of this type explain successfully sharp peaks in the diffraction patterns obtained by scattering of electrons, X-rays and neutrons.