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Determinants of Laplacians on Surfaces of Finite Volume

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Determinants of the Laplace and other elliptic operators on compact manifolds have been an object of study for many years (see [MP, RS, Vor]). Up until now, however, the theory of determinants has not been extended to non-compact situations, since these typically involve a mixture of discrete and continuous spectra. Recent advances in this theory, which are partially motivated by developments in mathematical physics, have led to a connection, in the compact Riemann surface case, between determinants of Laplacians on spinors and the Selberg zeta function of the underlying surface (see [DP, Kie, Sar, Vor]).

Our purpose in this paper is to introduce a notion of determinants on non-compact (finite volume) Riemann surfaces. These will be associated to the Laplacian Δ shifted by a parameter $s(1-s)$, and will be defined in terms of a Dirichlet series $\zeta(w,s)$ which is a sum that represents the discrete as well as the continuous spectrum. It will be seen to be regular at $w=0$, and our main theorem (see Sect. 1) will express $\exp\left(-\frac{\partial}{\partial w}\zeta(w,s)\Big|_{w=0}\right)$ as the Selberg zeta function of the surface times the appropriate Γ-factor.

1.

Let $M=\Gamma\backslash\mathbf{H}$ be a non-compact, finite volume surface obtained as the quotient of the upper half plane $\mathbf{H}$ by a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbf{R})$. For simplicity we assume that Γ has no fixed points. Let χ be a unitary character of Γ. We consider the spectral problem

$$\Delta f+\lambda f=0\,,\quad f(\gamma z)=\chi(\gamma)f(z)\quad(\gamma\in\Gamma, z\in\mathbf{H})\,,\quad \int_M |f(z)|^2dz<\infty\,. \tag{1.1}$$

Here $\Delta=y^2\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}\right)$ is the Laplacian of $\mathbf{H}$. In addition to a discrete spectrum $0\leqq\lambda_0\leqq\lambda_1\leqq\ldots$, this set-up gives rise to a continuous spectrum as well, as we now

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