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Generating Difference Equations with the Darboux Transformation

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Abstract. Starting with a system satisfying the so-called "bispectral property," a method is given to generate other systems with still higher-order recursion relations. The usefulness of the method is checked for the classical examples.

1. Introduction

For many well-known functions arising as eigenfunctions of ordinary differential equations, an important and common additional property is the existence of a recursion relation in the spectral parameter. This will always happen for orthogonal polynomials, but in practice it is also found for virtually anything with a name, as examination of books on the subject of classical functions will attest. For the purpose of practical computation, this property is beyond price.

In [5], Duistermaat and Grünbaum classified all sets of eigenfunctions satisfying differential operators in two different independent variables, the bispectral property. The partially discrete version of this, to classify all differential operators whose eigenfunctions also satisfy recursion relations, is a problem that remains open (see [7]). In this paper, the question is approached from a slightly different angle: Starting with a differential operator whose eigenfunctions also satisfy a recursion relation, a constructive method is given to generate other differential operators with the same property. This give hope of solving the classification problem in a different way: by showing that all solutions are generated from a small number of primitives.

1.1. Notation. We start with a quintuple $(L, B, \lambda, \Theta, \phi)$ satisfying

$$L\phi_n(x) = \lambda_n \phi_n(x) \tag{1}$$

and

$$(B\phi)_n \equiv \sum_{i=-N}^{N} b_i^{(n)} \phi_{n+i}(x) = \Theta(x)\phi_n(x). \tag{2}$$

Here $\partial \equiv d/dx \cdot L \equiv \partial^2 + V(x)$ is a formal differential operator in one variable x, and B is a difference operator in n. The formula (2) is an example of a *recursion relation*, since it provides a way to generate ϕ_{n+N} given the ϕ's for smaller n. In this paper, the eigenvalue Θ is assumed to be independent of n, and the b_i's independent of x. n will range over the integers.