Commun. Math. Phys. 119, 153–176 (1988)

Communications in
Mathematical
Physics

Existence of Excited States for a Nonlinear Dirac Field

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Abstract. We prove the existence of infinitely many stationary states for the following nonlinear Dirac equation

$$i\gamma^\mu \partial_\mu \psi - m\psi + (\bar{\psi}\psi)\psi = 0\,.$$

Seeking for eigenfunctions splitted in spherical coordinates leads us to analyze a nonautonomous dynamical system in $\mathbf{R}^2$. The number of eigenfunctions is given by the number of intersections of the stable manifold of the origin with the curve of admissible datum. This proves the existence of infinitely many stationary states, ordered by the number of nodes of each component.

1. Introduction

We study the existence of stationary states for the following nonlinear Dirac equation

$$i \sum_{\mu=0}^{3} \gamma^\mu \partial_\mu \psi - m\psi + F(\bar{\psi}\psi)\psi = 0\,. \tag{1.1}$$

The notation is the following. ψ is defined on $\mathbf{R}^4$ with values in $\mathbf{C}^4$, $\partial_\mu = \partial/\partial x_\mu$, m is a positive constant, $\bar{\psi}\psi = (\gamma^0\psi, \psi)$, where $(\,,)$ is the usual scalar product in $\mathbf{C}^4$, and the γ^μ's are the 4×4 matrices of the Pauli-Dirac representation, given by

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad \text{and} \quad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix} \quad \text{for} \quad k = 1, 2, 3\,,$$

where

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Finally, $F: \mathbf{R} \to \mathbf{R}$ models the nonlinear interaction.

We are interested in stationary states, or localized solutions of (1.1), that is solutions ψ of the form $\psi(t,x) = e^{i\omega t}\varphi(x)$, where $t = x_0$ and $x = (x_1, x_2, x_3)$. In