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# Linear Cellular Automata and Recurring Sequences in Finite Fields

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**Abstract.** A one-dimensional linear cellular automaton with periodic boundary conditions consists of a lattice of sites on a cylinder evolving according to a linear local interaction rule. Limit cycles for such a system are studied as sets of strings on which the rule acts as a shift of size $s/h$; i.e., each string in the limit cycle cyclically shifts by $s$ sites in $h$ iterations of the rule. For any given rule, the size of the shift varies with the cylinder size $n$. The analysis of shifts establishes an equivalence between the strings of values appearing in limit cycles for these automata, and linear recurring sequences in finite fields. Specifically, it is shown that a string appears in a limit cycle for a linear automaton rule on a cylinder size $n$ iff its values satisfy a linear recurrence relation defined by the shift value for that $n$. The rich body of results on recurring sequences and finite fields can then be used to obtain detailed information on periodic behavior for these systems. Topics considered here include the inverse problem of identifying the set of linear automata rules for which a given string appears in a limit cycle, and the structure under operations (such as addition and complementation) of sets of limit cycle strings.

## 1. Introduction

A central feature of cellular automata with periodic boundary conditions is their generation of limit cycle behavior. In one dimension, an automaton of this type may be viewed as a lattice of sites on a cylinder of specified size $n$ evolving according to a local interaction rule of the form

$$x_i^{t+1} = f(x_{i-r}^t, \ldots, x_i^t, \ldots, x_{i+r}^t), \quad f: F_q^{2r+1} \to F_q \tag{1.1}$$

together with the condition

$$x_i^t = x_j^t, \quad i \equiv j \bmod n,$$

for all $i$ and $t$. The finiteness and deterministic nature of the system then imply that, for arbitrary initial conditions, the spatial sequences generated eventually become periodic. In the case that the function $f$ in (1.1) is linear, Martin, Odlyzko, and Wolfram [1] have provided an extensive study of transience, limit cycle