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# Trapping and Cascading of Eigenvalues In the Large Coupling Limit

F. Gesztesy [1,4,*,**], D. Gurarie [1,***], H. Holden [2], M. Klaus [3], L. Sadun [1], B. Simon [1,**], and P. Vogl [4]

[1] Division of Physics, Mathematics, and Astronomy, California Institute of Technology, Pasadena, CA 91125, USA

[2] Institute of Mathematics, University of Trondheim, N-7034 Trondheim-NTH, Norway

[3] Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA

[4] Institute for Theoretical Physics, University of Graz, A-8010 Graz, Austria

**Abstract.** We consider eigenvalues $E_\lambda$ of the Hamiltonian $H_\lambda = -\Delta + V + \lambda W$, $W$ compactly supported, in the $\lambda \to \infty$ limit. For $W \geqq 0$ we find monotonic convergence of $E_\lambda$ to the eigenvalues of a limiting operator $H_\infty$ (associated with an exterior Dirichlet problem), and we estimate the rate of convergence for 1-dimensional systems. In 1-dimensional systems with $W \leqq 0$, or with $W$ changing sign, we do not find convergence. Instead, we find a cascade phenomenon, in which, as $\lambda \to \infty$, each eigenvalue $E_\lambda$ stays near a Dirichlet eigenvalue for a long interval (of length $O(\sqrt{\lambda})$) of the scaling range, quickly drops to the next lower Dirichlet eigenvalue, stays there for a long interval, drops again, and so on. As a result, for most large values of $\lambda$ the discrete spectrum of $H_\lambda$ is close to that of $H_\infty$, but when $\lambda$ reaches a transition region, the entire spectrum quickly shifts down by one. We also explore the behavior of several explicit models, as $\lambda \to \infty$.

## 1. Introduction

In quantum mechanics one frequently encounters Hamiltonians of the form $H_\lambda = H_0 + \lambda W$, where $H_0$ describes a well-understood system (the "background" or "free" Hamiltonian), $W$ describes any of various interactions in the system (e.g. interacting particles, and external fields, etc.), and $\lambda$ (the "coupling constant") measures the strength of the interaction $W$. In this paper we consider Hamiltonians

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